Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

```
Clear["Global`*"]
```

Problems related to theorems 1 and 2.
1 - 4 Verify theorem 1 for the given F[z], z_0, and circle of radius 1.

1. $(z+1)^3$, $z_0 = \frac{5}{2}$

```
Clear["Global`*"]
```

$F[z\_] = (z+1)^3$

$(1+z)^3$

I can put the problem details in the form of theorem 1

$\frac{1}{2\pi}\text{Integrate}\left[\left(\frac{5}{2}+1+e^{i\alpha}\right)^3, \{\alpha, 0, 2\pi\}\right]$

$\frac{343}{8}$

and then compare with direct calculation of the specified z_0, the theorem conclusion.

$F\left[\frac{5}{2}\right]$

$\frac{343}{8}$

3. $2z^4$, $z_0 = 4$

```
Clear["Global`*"]
```

$F[z\_] = (2z)^4$

$16z^4$

The problem function is put into the form of the theorem 1 statement

$\frac{1}{2\pi}\text{Integrate}\left[\left(2\left(4+e^{i\alpha}\right)\right)^4, \{\alpha, 0, 2\pi\}\right]$

4096

and compared with the direct calculation of the proposed z_0.

```
F[4]
```

```
4096
```

5. Integrate Abs[z] around the unit circle. Does the result contradict theorem 1?

```
Clear["Global`*"]
```

In this case $z_0=0$

```
F[z_] = Abs[z]
Abs[z]
```

```
Integrate[Abs[0 + e^(i α)], {α, 0, 2 π}]
2 π
```

```
F[0]
0
```

Theorem 1 does not seem to hold for the absolute value function. I had to look at the text answer, which points out that the absolute value function is not analytic, therefore not eligible for application of theorem 1.

7 - 9 Verify (3) in theorem 2 for the given Φ[x,y], (x_0, y_0), and circle of radius 1.

7. (x - 1) (y - 1), (2, -2)

```
Clear["Global`*"]
```

Numbered line (3) on p. 782 goes

$$\Phi[x_0, y_0] = \frac{1}{\pi r_0^2} \int_0^{r_0} \int_0^{2\pi} \Phi[x_0 + r \cos[\alpha], y_0 + r \sin[\alpha]]\, r \, d\alpha \, dr$$

which is what I need to verify for the problem function and specified point. The expression of numbered line (3), p. 782, is repeated below. I can note that for this problem, $r_0 = 1$, $\Phi[x, y] = (x-1)(y-1)$, and $\{x_0, y_0\} = \{2, -2\}$.

So simplifying the function equation,

$$\Phi[\{x\_, y\_\}] = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} (x-1)(y-1)\, dy\, dx$$

$$1 - \pi$$

and comparing with the result of numbered line (3)

```
Φ[{2, -2}]
```

$$1 - \pi$$

9. x + y + x y, (1, 1)

Repeating the matter of numbered line (3) on p. 782:

$$\Phi[x_0, y_0] = \frac{1}{\pi r_0^2}\int_0^{r_0}\int_0^{2\pi}\Phi[x_0 + r\,\mathrm{Cos}[\alpha], y_0 + r\,\mathrm{Sin}[\alpha]]\; r\, d\alpha\, dr$$

which is what I need to verify for the problem function and specified point. The expression of numbered line (3) is repeated below. I can note that $r_0 = 1$, $\Phi[x, y] = x + y + x y$, and $\{x_0, y_0\} = \{1, 1\}$.

```
Clear["Global`*"]
```

Including the problem function,

$$\Phi[\{x\_, y\_\}] = \frac{1}{\pi}\int_0^1\int_0^{2\pi}(x + y + xy)\,dy\,dx$$

```
1 + 2 π + 2 xy
```

and calculating the result of the given point

```
Φ[{1, 1}]
```

```
1 + 2 π + 2 xy
```

13 - 17 Maximum modulus

Find the location and size of the maximum of Abs[F[z]] in the unit disk Abs[z] ≤ 1.

13. F[z] = Cos[z]

```
Clear["Global`*"]
```

```
FindMaximum[{Abs[Cos[x + i y]], {x, y} ∈ Disk[{0, 0}, 1]}, {x, y}]
```

$\{1.54308, \{x \to -8.1408\times10^{-9}, y \to 1.\}\}$

```
FindMaximum[{Abs[Cos[z]], -1 ≤ z ≤ 1}, {z}]
```

$\{1., \{z \to -1.84375\times10^{-8}\}\}$

The answer in green agrees with the text answer. Mathematica in this case came up with the answer without display of hyperbolic trig functions. Even though I am looking for the modulus, expressing the search as monolithic z does not work.

15. F[z] = Sinh[2 z]

```
Clear["Global`*"]
```

The first try does not work in obtaining the maximum value of z.

```
FindMaximum[{Abs[ Sinh[2 (x + ⅈ y)]], {x, y} ∈ Disk[{0, 0}, 1]}, {x, y}]
{2.03809, {x → 0.669846, y → -0.742498}}
```

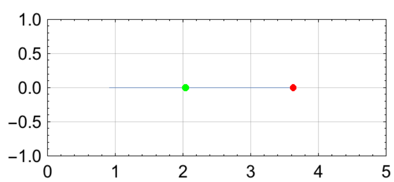
I see that this problem is very easy using just z, probably because the answer is on the x-axis.

```
FindMaximum[{Abs[Sinh[2 z]], -1 ≤ z ≤ 1}, {z}]
```

```
{3.62686, {z → 1.}}
```

However, if I want to work with x + ⅈ y it is harder to get what I want. First, a plot

```
d2 = DiscretizeRegion@ImplicitRegion[x^2 + y^2 ≤ 1, {x, y}];
ParametricPlot[ReIm[Abs[ Sinh[2 (x + ⅈ y)]]],
 {x, y} ∈ d2, PlotRange → {{0, 5}, {-1, 1}},
 Frame → True, ImageSize → 200, AspectRatio → Automatic,
 Axes → False, PlotStyle → Thick, GridLines → Automatic,
 Epilog → {{Red, PointSize[0.02], Point[{3.62686, 0}]},
   {Green, PointSize[0.02], Point[{2.03809, 0}]}}]
```



Showing that Mathematica can come up with the right answer graphically, when making use of the complex plane. What I found, I believe, is that with FindMaximum it helps greatly if I put in the starting guesses for x and y, as

```
FindMaximum[{Abs[ Sinh[2. * (x + ⅈ y)]], {x, y} ∈ d2},
 {{x, 0}, {y, 0}}, AccuracyGoal → 20, PrecisionGoal → 18]
```

FindMinimum::eit:
The algorithm does not converge to the tolerance of 4.806217383937354`*^-6 in 500 iterations. The best estimated solution, with feasibility residual, KKT residual, or complementary residual of {630.219, 77.7632, 277.564}, is returned. ≫

FindMinimum::eit:
The algorithm does not converge to the tolerance of 4.806217383937354`*^-6 in 500 iterations. The best estimated solution, with feasibility residual, KKT residual, or complementary residual of {7.45092, 132.609, 1.27536}, is returned. ≫

FindMaximum::eit:
The algorithm does not converge to the tolerance of $\frac{1}{100000000000000000000}$ in 500 iterations. The best estimated solution, with feasibility residual, KKT residual, or complementary residual of $\{4.97687\times10^{-12}, 5.75614\times10^{-6}, 4.42274\times10^{-12}\}$, is returned. ≫

$\{3.62686, \{x \to 1., y \to -3.08685\times10^{-7}\}\}$

Producing, after a little trouble, the answer I wanted. What is troubling is that I did not see a clue that the original answer was defective. It was only by looking at the text answer that I realized that the easy way was wrong.

17. $F[z] = 2 z^2 - 2$

```
Clear["Global`*"]
```

```
FindMaximum[{Abs[2 (x + i y)^2 - 2], {x, y} ∈ Disk[{0, 0}, 1]}, {x, y}]
```

$\{4.,\ \{x \to 1.48588 \times 10^{-8},\ y \to 1.\}\}$

```
Solve[Abs[2 z^2 - 2] == 4, z]
```

Solve::ifun:
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

$\{\{z \to -i\},\ \{z \to i\},\ \{z \to -\sqrt{3}\},\ \{z \to \sqrt{3}\}\}$

The green cell finds the maximum value sought by the problem. The yellow cell gives a suggestion of $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, since for some reason the text answer has z in angular measure.